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Glafka 2004: Some Remarks on the Role of Complex Numbers in Quantum Theory

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We consider the issue of whether complex numbers are necessary in the formulation of quantum theory. We argue that their introduction is not forced on us by the dynamics, but it is absolutely necessary in order to incorporate the observable properties of the geometric phases that appear in interference experiments. This remark provides the motivation for the construction of a histories-based axiomatic scheme for quantum theory, in which phases are considered as primitive elements, on equal footing with probabilities. This scheme reduces to standard quantum theory for systems characterised by a background causal structure–however it may lead to a different description in the domain of quantum gravity.

KEY WORDS: quantum mechanics; decoherent histories; quantum phase.

1. INTRODUCTION

The fact that the formalism of quantum theory is phrased in terms of complex numbers (rather than real ones) has always been a matter of speculation. Is the use of complex numbers a mere mathematical convenience or does it signify something deeper about the structure of quantum mechanics? It is often argued that the introduction of complex numbers is forced upon us by the dynamical law, namely Schrödinger's equation

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi + V(x)\psi := \hat{H}\psi. \tag{1}$$

The presence of the i in the right-hand terms implies that the wave function cannot be in general, real-valued, and for this reason it has to be postulated real-valued from the beginning.

This argument, however, is not satisfactory. A complex vector space can easily be interpreted as a real one, and a unitary transformation as an orthogonal

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one. One may split a complex wave-function into a real and imaginary part

$$\psi \to \begin{pmatrix} \psi_R \\ \psi_I \end{pmatrix}, \tag{2}$$

and introducing the real-valued matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we may write Schrödinger's equation solely in terms of real numbers

$$J \frac{\partial}{\partial t} \begin{pmatrix} \psi_R \\ \psi_I \end{pmatrix} = \hat{H} \begin{pmatrix} \psi_R \\ \psi_I \end{pmatrix}. \tag{3}$$

Hence, the evolution law alone does not necessitate the introduction of complex numbers. This may be seen, in particular, in Bohmian mechanics, in which the evolution equation is cast is such a form that the complex structure of the Hilbert space can be ignored and is only employed for notational simplicity.

The physical principle that makes the introduction of complex numbers necessary is Born's probability interpretation of the wave function: the probability density $|\psi(x)|^2$ is invariant under the $U(1)$ transformation of ψ: $\psi \to \psi e^{i\phi}$. Out of all *a priori* conceivable ways to construct a probability density out of the wave function, quantum theory selects one that is characterised by an additional symmetry. We shall argue that the presence of this symmetry is the reason that complex numbers are necessary (if not indispensable) in quantum theory.

Before proceeding further in this discussion we would like to remind the reader of a relevant point. It is possible to formulate quantum mechanics in a real Hilbert space: however, if one demands that the Heisenberg uncertainty relations arise naturally in the theory, it is necessary that one introduces a complex structure on the Hilbert space (Stueckelberg, 1960), effectively leading to a complex Hilbert space.

2. GEOMETRIC PHASES

The above arguments suggest that the necessity of using complex numbers in quantum theory arises as a consequence of the special probabilistic structure of quantum theory, namely the fact that the evolution law is not linear with respect to the probabilities, but only with respect to the system's wave function. One may respond, however, that irrespective of the way probabilities are defined in quantum theory, at the end of the day all physical predictions are phrased in terms of real numbers (e.g. event probabilities, mean values of physical observables, or scattering cross-sections). The complex structure of quantum theory is not manifested at the level of physical predictions.

The answer to this objection is that complex numbers arise from the $U(1)$ symmetry characteristic of the quantum probability law, which is manifested experimentally in the form of geometric phases. To see this one may consider the two slit experiment. A source is prepared to emit particles in a given state ψ. The particle beam is directed towards a wall with two slits in it, while a screen in which the particles are detected is placed behind the two slits. If both slits are open then the intensity of the particles recorded on the screen will exhibit a periodicity, that is we shall determine an interference pattern.

The key point is that if we act on the particle beams as they cross the two slits, the shape of the interference pattern will change. A typical example is the Bohm-Aharonov effect (Aharonov and Bohm, 1959). Let us assume that in the original two-slit experiment the state after the particles pass through the slits is $\psi = \frac{1}{\sqrt{2}}(\psi_L + \psi_R)$. The presence of a solenoid behind the slits will effectively transform ψ into $\frac{1}{\sqrt{2}}(\psi_L + e^{i\theta}\psi_R)$, in terms of a phase $\theta = q\Phi$, with q the particles' charge and Φ the solenoid's flux. Hence if we compare the distribution of particles detected in the screen in the experiment without the solenoid $|\psi_L|^2 + |\psi_R|^2 + 2Re\psi_R^*\psi_L$ with that in the presence of the solenoid $|\psi_L|^2 + |\psi_R|^2 + 2Ree^{i\phi}\psi_R^*\psi_L$ we shall see a Φ dependent translation of the peaks and lows of the interference pattern. The important point is that if we repeat this experiment for different values of the flux, we shall see that the dependence of the particle intensity on Φ is periodic, since Φ only appears in the form of $e^{iq\Phi}$. In other words, the change induced to the interference pattern by changes of the external parameters is subject to a $U(1)$ symmetry (corresponding to the Bohm-Aharonov phase).

The Bohm-Aharonov phase is a special case of Berry's geometric phase (Berry, 1984); in the most general case it is defined as the holonomy of the natural $U(1)$ connection on the projective space of a complex Hilbert space (Anandan and Aharonov, 1990; Simon, 1983). In this sense, the observation of geometric phases (with their periodic behaviour) provides the most direct experimental evidence that the complex numbers (or equivalently the U(1) symmetry) is a fundamental ingredient of quantum theory.

It is important to emphasise that the geometric phase is a *statistical* object: it is measured in terms of an interference pattern, which is present only when a large number of particles (corresponding to a statistical ensemble are left to interfere). If we carried out the experiment with single particle, there would be nothing to measure.

A very general type of geometric phase that will prove particulary relevant to our later discussion is the *Pancharatnam* phase (Pancharatnam, 1956). It corresponds to the argument of the inner product between two states $\phi\rangle$ and $|\psi\rangle$. It can be determined through the following procedure

1. We prepare two systems in the states $|\psi\rangle$ and $|\phi\rangle$.

2. We perform on the beam corresponding to $|\psi\rangle$ the operation $|\psi\rangle \to e^{i\chi}|\psi\rangle$ for a controlled value of χ.

3. We interfere the two beams to construct the beam $|f\rangle = |\phi\rangle + e^{i\chi}|\psi\rangle$ and measure its intensity $I = \langle f|f\rangle$.

4. We repeat the experiment for the range of all values of χ and construct the function $I(\chi)$ giving the intensity of the measured beam as a function of the external parameter χ. This equals

$$I(\chi) = 2|\langle\psi|\phi\rangle|\cos(\chi - \arg\langle\psi|\phi\rangle) \tag{4}$$

5. $I(\chi)$ takes its maximum value for $\chi = \arg\langle\psi|\phi\rangle$. This value of χ is the Pancharatnam phase between the two beams.

This procedure to measure the Pancharatnam phase has been performed in neutron interferometry (Wagh *et al.*, 1998). The difficult part is to perform step 2, i.e. to have a controlled way to change the phase of an individual quantum state. This can be achieved if (for instance $\psi\rangle$ is an eigenstate of a Hamiltonian (so that the phase only depends on the number of periods the beam is left before interference).

An important feature of the geometric phases is that they cannot be absolutely defined. They are relative; as can be seen from the consideration of the Aharonov-Bohm experiment they can only be determined if we interfere two beams with different past history (corresponding to the particles passing through either of the two slits). This strongly suggests that the basic features of geometric phases and consequently the role of complex numbers in quantum theory will be more strongly manifested in a formalism based upon *histories*.

3. CONSISTENT HISTORIES

The consistent histories approach to quantum theory was developed by Griffiths (1984), Omnés (1988, 1994), Gell-Mann and Hartle (1990, 1993), Hartle (1993). The basic object is a history, which corresponds to properties of the physical system at successive instants of time. A discrete-time history α will then correspond to a string $\hat{P}_{t_1}, \hat{P}_{t_2}, \ldots \hat{P}_{t_n}$ of projectors, each labelled by an instant of time. From them, one can construct the class operator

$$\hat{C}_\alpha = \hat{U}^\dagger(t_1)\hat{P}_{t_1}\hat{U}(t_1)\ldots\hat{U}^\dagger(t_n)\hat{P}_{t_n}\hat{U}(t_n) \tag{5}$$

where $\hat{U}(s) = e^{-i\hat{H}s}$ is the time-evolution operator. The probability for the realisation of this history is

$$p(\alpha) = Tr\left(\hat{C}_\alpha^\dagger\hat{\rho}_0\hat{C}_\alpha\right), \tag{6}$$

where $\hat{\rho}_0$ is the density matrix describing the system at time $t = 0$.

But this expression does not define a probability measure in the space of all histories, because the Kolmogorov additivity condition cannot be satisfied: if α and β are exclusive histories, and $\alpha \vee \beta$ denotes their conjunction as propositions, then it is not true that

$$p(\alpha \vee \beta) = p(\alpha) + p(\beta). \tag{7}$$

The histories formulation of quantum mechanics does not, therefore, enjoy the status of a genuine probability theory.

However, an additive probability measure *is* definable, when we restrict to particular sets of histories. These are called *consistent sets*. They are more conveniently defined through the introduction of a new object: the decoherence functional. This is a complex-valued function of a pair of histories given by

$$d(\alpha, \beta) = Tr\left(\hat{C}^{\dagger}_{\beta}\hat{\rho}_0\hat{C}_{\alpha}\right). \tag{8}$$

A set of exclusive and exhaustive alternatives is called consistent, if for all pairs of different histories α and β, we have

$$Re\ \ d(\alpha, \beta) = 0. \tag{9}$$

In that case one can use equation (6) to assign a probability measure to this set.

While the consistent histories interpretation refers to properties of closed individual systems, the same formalism can be applied in an operational setting similar to that of the Copenhagen interpretation. In that context the expression (6) refers to the probability corresponding to a multi-time measurement scheme in which we determine whether the properties $\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \ldots, \hat{\alpha}_{t_n}$ are satisfied at times $t_1, t_2, \ldots, t_n$.

If we consider an operational interpretation of the histories formalism, we may easily that the off-diagonal elements of the decoherence functional may be in principle expressed as a special case of the Pancharatnam phase.

Let us assume we have a source S preparing a beam of particles in a state $|\psi\rangle$. After exiting S the particles enter a beam splitter B.S. One subbeam then enters a sequence of filters $\alpha_{t_1} \ldots \alpha_{t_n}$ and the other a sequence of filters $\beta_{t'_1} \ldots \beta_{t'_m}$, before they are recombined at C. the beam then propagates to a screen, where its intensity is measured. We repeat this experiment many times, but at each time the second component of the split beam has to pass through P.O. which performs the operation of phase change $|\psi\rangle \rightarrow e^{i\chi}|\psi\rangle$. We, thus, get a function $I(\chi)$, whose maximum determines a phase that is the argument of the value of the decoherence functional between the histories $(\alpha_{t_1} \ldots \alpha_{t_n})$ and $(\beta_{t'_1}, \ldots, \beta_{t'_m})$. The modulus of the phase of the decoherence functional can easily be determined by the maximum value of I (see Anastopoulos (2001, 2003) for details).

In fact, it can be shown that the decoherence functional contains all information that can be obtained from experiments measuring either probabilities or relative phases (Anastopoulos, 2003). The complex-valued temporal correlation functions of quantum theory can in principle be determined by the measurement of a sufficiently large number of interference phases as in the experiment described above. In effect, if one can determine the off-diagonal elements of a decoherence functional between a history $\alpha_{ij} = (P_i, t_1; P_j, t_2)$ and the trivial history $\beta = 1$, then for the observable $A = \sum_i a_i P_i$ the time-ordered two-point function will read

$$\langle A_{t_1} A_{t_2} \rangle_Q = \sum_{ij} d(\alpha_{ij}, 1), \tag{10}$$

4. AN AXIOMATIC SCHEME FOR QUANTUM THEORY

The discussion above on the importance of geometric phases in demonstrating the role of complex numbers in quantum theory motivates us to construct an axiomatic framework for quantum theory, in which the quantum phases plays the role of a primitive ingredient. As we have indicated before, such a scheme must be fundamentally based on histories.

Most axiomatic schemes for quantum theory introduce complex numbers at the level of the basic physical observables: fir example the standard formalism introduces them in the postulate f a complex Hilbert space; the C^*-algebra formalism postulates an algebra over the field of complex numbers and then introduces and implements Born's rule by the introduction of an additive probability measure over the algebra. In these schemes the introduction of the complex numbers goes hand in hand with the consideration of *non-classical* observables, namely through the representation of physical observables by non-commuting algebraic objects.

Remarkably, the introduction of quantum phases as a primitive ingredient in an axiomatic formulation allows one to obtain all predictions of quantum theory from a scheme with classical observables. In other words, if one postulates that the quantum formalism predicts not only probabilities but also the values of relative phases between different histories, then it is not necessary to introduce non-commutative objects as fundamental quantum entities.

We shall next proceed to describe this scheme, which is essentially an adaptation of the basic axioms for histories as set out by Hartle (1993), Isham (1994), Isham and Linden (1994).

4.1. Observables

At the level of observables, the structure of our theory is identical with that of classical probability theory. That is, we assume the existence of a space Ω of elementary alternatives. A point of Ω corresponds to the most precise information one can extract from a measurement of the quantum system. Note, that at this level we do not distinguish, whether Ω refers to properties of a systems at one moment of time or to histories.

The space Ω has to be equipped with some additional structure. In general, a measurement will yield some information stating that the system was found in a given subset of Ω. But not all subsets of Ω are suitable to incorporate measurement outcomes. For instance, when we consider position it is physically meaningless to consider the subset of rational values of position (with respect to some unit). One, therefore needs to choose a family of subsets $\mathcal{C}$ of Ω, that correspond to the coarse-grained information we can obtain about the physical systems. These sets are often called *events*. The family $\mathcal{C}$ containing the events has to satisfy some natural mathematical conditions: in mathematical terms $\mathcal{C}$ has to be a σ-field. The relevant conditions are the following

1. $\Omega \in \mathcal{C}$.

2. $\emptyset \in \mathcal{C}$.

3. If $A \in \mathcal{C}$, then $\Omega - A \in \mathcal{C}$.

4. If $A, B \in \mathcal{C}$, then $A \cup B \in \mathcal{C}$ and $A \cap B \in \mathcal{C}$.

5. For countably many $A_n \in \mathcal{C}$, $n = 1, 2, \ldots, \cup_{n=1}^{\infty} \in \mathcal{C}$.

Equipping Ω with a σ-field turns it into a *measurable space*. An observable is a measurable function on Ω, namely one that preserves the σ-field structure. We shall denote the space of measurable functions to in as $F(\Omega)$.

We will be interested in the case of histories, namely when the sample space Ω is a path space, namely a subset of $\times_t \Gamma_t$, where Γ_t is the space of elementary alternatives at a specific moment of time. Usually we shall consider that Γ_t is isomorphic to a space Γ for all values of t.

4.2. Probabilities

In the axiomatic scheme we present here the basic observables are classical, not fundamentally different from the ones employed in classical probability theory. The quantum character is brought in by the introduction of phases as a primitive ingredient (something that also results to the probability measure being non-additive). This is achieved by the postulate of a decoherence functional, whose diagonal elements correspond to probabilities and the off-diagonal to the (in principle measurable) Pancharatnam phases between the histories. Note that the present axiomatic scheme is phrased within the Copenhagen interpretation, namely both probabilities and phases are assumed to refer to the outcomes of concrete experimental situations.

A *decoherence functional* D is a map from $\mathcal{C} \times \mathcal{C} \to \mathbf{C}$, such that the following conditions are satisfied

B1. *Null triviality:* For any $A \in \mathcal{C}$, $D(\emptyset, A) = 0$.
In terms of our interpretation of the off-diagonal elements of the decoherence functional as corresponding to Pancharatnam phases, there can be no phase measurement if one of the two beams that have to be interfered is absent.

B2. *Hermiticity:* For $A, B \in \mathcal{C}$, $D(B, A) = D^*(A, B)$.
Clearly the phase difference between two histories becomes opposite if we exchange the sequence, by which these histories are considered.

B3. *Positivity:* For any $A \in \mathcal{C}$, $D(A, A) \geq 0$.
This amounts to the fact that the diagonal elements of the decoherence functional are interpreted as probabilities (albeit non-additive). Operationally probabilities are defined by the number of times a particular event occurred in the ensemble and by definition they can only be positive.

B4. *Normalisation:* $D(\Omega, \Omega) = 1$.
Clearly, if no measurement takes place the intensity of the beam would never change.

B5. *Additivity:* If $A, B, C \in \mathcal{C}$ and $A \cap B = \emptyset$, then $D(A \cup B, C) = D(A, C) + D(B, C)$.
There is no intuitive operational reason, why this should be the case. This property is equivalent to the superposition principle of quantum theory and we can consider that it is forced upon us by experimental results. Of course, this is the property that makes the decoherence functional the natural object to use.

B6. *Boundedness:* For all $A, B \in \mathcal{C}$, $|D(A, B)| \leq 1$.

4.3. Equivalence to Standard Quantum Mechanics

The set of axioms above is equivalent to standard quantum theory. This equivalence is stated in the propositions 1 and 2 that follow–for detailed proofs see Ref. Anastopoulos (2003).

Proposition 1. *All quantum mechanical systems for which a classical phase space may be identified may be described by the above axiomatic scheme.*

We choose the space Γ as the classical phase space. If H the single-time quantum mechanical Hilbert space, introduce coherent states $|z\rangle$, $z \in \Gamma$.

The decoherence functional takes the following value for two- discrete time histories corresponding to the projectors $|z_{t_1}\rangle\langle z_{t_1}|, |z_{t_2}\rangle\langle z_{t_2}|, \ldots |z_{t_n}\rangle\langle z_{t_n}|$ and $|z'_{t'_1}\rangle\langle z'_{t'_1}|, |z'_{t'_2}\rangle\langle z'_{t'_2}|, \ldots |z'_{t'_m}\rangle\langle z'_{t'_m}|$

$$D_{z_0}(z_1, t_1; z_2, t_2; \ldots; z_n, t_n | z'_1, t'_1; z'_2, t'_2; \ldots; z'_m, t'_m) = \langle z'_m | e^{-iH(t'_m - t_n)} | z_n \rangle \langle z_n | e^{-iH(t_n - t_{n-1})} | z_{n-1} \rangle \ldots \langle z_2 | e^{-iH(t_2 - t_1)} | z_1 \rangle \langle z_1 | e^{-iH(t_1 - t_0)} | z_0 \rangle \langle z_0 | e^{-iH(t_0 - t'_1)} | z'_1 \rangle \langle z'_1 | e^{-iH(t'_1 - t'_2)} | z'_2 \rangle \ldots \langle z'_{m-1} | e^{-iH(t'_{m-1} - t'_m)} | z'_m \rangle, \tag{11}$$

where in the above equation the initial state is assumed to be a coherent sate $|z_0\rangle$. At the continuous limit (for differentiable paths)

$$D[z(\cdot), z'(\cdot)] \sim e^{iS[z(\cdot)] - iS[z'(\cdot)]}, \tag{12}$$

where $S = \int dt(\langle z_t | \dot{z}_t \rangle - \langle z_t | \hat{H} | z_t \rangle$ is the classical phase space action.

It is easy to see from the equation above that if the paths $z(\cdot)$ and $z'(\cdot)$ have the same endpoints, the form a loop C and D equals the holonomy of a $U(1)$ connection A on the extended phase space $\Gamma_{ext} = \Omega \times \Omega$ ($\Omega = \times_t \Gamma_t$) of the classical system which includes the time variable $t \in \mathbf{R}$ as a variable. (For a particle at a line this connection is $A = pdq - H(p, q)dt$).

It follows that any quantum mechanical system, in which coherent states can be defined defines a process on a phase space Γ that satisfies the axioms 1–6.

We should note at this point that the decoherence functional D, viewed as a functional on the space of paths Ω has support on non-differentiable paths (the so-called cylinder sets). For this reason it is not fully specified by its values on the continuous-time paths. An additional structure is necessary, which is incorporated in the coherent states propagator $\langle z | e^{-i\hat{H}t} | z' \rangle$. This essentially corresponds to the coarse-graining operation. The reason is the following. To consider coarse-grained alternatives we must essentially sum over paths; this involves the definition of a measure on Ω. The choice of this measure is dependent upon the properties of the coherent state propagator (Anastopoulos, 2003; Anastopoulos and Savvidou, 2003).

The axioms 1–6 are more general than the usual axioms defining standard quantum theory. If, however, we assume that the system also satisfies the Markov property, it is possible to prove an equivalence. The Markov property roughly states that if the state of the system (i.e the restricted decoherence functional at a moment of time) is completely specified, then it contains sufficient information to determine the state of the system at any subsequent moment of time. The exact mathematical phrasing of this condition can be shown to be equivalent to the statement (Anastopoulos, 2003) that the discrete-time decoherence functional satisfies the following factorisation condition

$$D(z_0, t_0; z_1, t_1; \ldots; z_N, t_N | z'_0, t_0; z_1, t_1; \ldots; z'_N, t_N) =$$

$$\upsilon(z_N, z'_N; t_N | z_{N-1}, z'_{N-1}; t_{N-1}) \ldots \upsilon(z_1, z'_1; t_1 | z_0, z'_0; t_0) \rho_0(z_0, z'_0), \quad (13)$$

in terms of an initial "state" at $t = 0$ and a propagator $\upsilon(z_1, z_2; t | z'_1, z'_2; t')$ that satisfies' the *quantum Chapman-Kolmogorov equation*:

$$\upsilon(z_1, z'_1; t | z_0, z'_0; s) = \int dz dz' \upsilon(z_1, z'_1; t | z, z'; s') \upsilon(z, z'; s' | z_0, z'_0; s). \quad (14)$$

We may then prove the following theorem

Proposition 2. *Assume we have a system satisfying axioms 1–6, with* $\Omega = \times_t \Gamma_t$ *that in addition satisfies the Markov property. If in addition*

i. the propagator is a smooth function of its arguments and the time entries,

ii. the process is time-homogeneous and time-reversible then we can reconstruct a quantum mechanical Hilbert space H *and the Heisenberg evolution equations, so that* $\upsilon_t(z_1, z_2, t | z'_1, , z'_2, t') = \psi_{t'-t}(z_1 | z'_1) \psi^*_{t'-t}(z_2 | z'_2)$, *with*

$$\psi_t(z|z) = \langle z | e^{-i\hat{H}t} | z' \rangle, \quad (15)$$

where $|z\rangle$ *is a set of coherent states on the Hilbert space* H.

Hence the axioms 1–6 allow a full reproduction of the usual formalism for standard quantum theory.

4.4. Further Remarks

The framework above does not solve any of the long-standing interpretational issues of quantum theory; indeed it is essentially cast in the form of the Copenhagen interpretation, the main difference being that phases are taken as primitive ingredients in addition to the probabilities. However, it serves to make two points. First, it is possible to describe quantum theory in terms of purely commutative observables and absorb all physical consequences of non-commutativity into the introduction of the quantum phase as an *irreducible* element of the formalism. Quantum logic is therefore not a necessary consequence of the empirical success of quantum theory.

Second, to obtain the structure of the Hilbert space necessitates the Markov condition, which presupposes a background causal structure. Hence, standard quantum mechanics arises as a consequence of the postulate of a background time. In absence of such a structure, as could be the case in quantum gravity *the*

Hilbert space is not necessary or perhaps not even natural. This suggests that a histories-based framework on quantum gravity may lead to very different results from those obtained by canonical quantisation—see the discussion in Savvidou (2004a,b). One such example where this idea may be tested is by attempting the construction of a quantum version of the growth processes for causal sets discussed in Rideout and Sorkin (2000) using the present axiomatic scheme.

The equivalence between the axioms 1–6 and standard quantum theory is based on the assumption that one may find a classical phase space for every known physical system that is described by quantum theory. While this is straightforward for particle systems and bosonic fields, this can not be easily identified for spinor fields, due to the presence of the canonical anti-commutation relations. Nonetheless, free spinor fields can be constructed by the Fock method from the Hilbert space of relativistic particles with half-integer spin, which can be fully described in a phase space language–see Souriau (1997) for the construction of the corresponding phase space. For the case of such particles the construction of a process satisfying the axioms 1–6 has been verified Anastopoulos (2004). It is based upon generalised coherent states of the Poincaré group $|X, I, J\rangle$, the arguments of which lie on an extended phase space consisting of a point of Minkowski spacetime X, a unit future-directed timelike 4-vector I (normalised 4-momentum) and a unit spacelike vector J (normalised Pauli-Lubanski vector), such that $I \cdot J = 0$.

The decoherence functional is constructed as in equation (11) and in the continuum limit corresponds to the following connection on Γ_{ext}

$$A = -MI_\mu dX^\mu - \frac{in}{2}[\lambda_A \epsilon^{AB} d\mu_B - \lambda^*_{A'} \epsilon^{A'B'} d\mu^*_{B'}],$$

where λ, μ are two-spinors corresponding to the null tetrad defined by $I + J$, $I - J$ respectively.

As a conclusion to this talk we would like to make an important remark about the way the U(1) symmetry and complex numbers are implemented in the axiomatic scheme we considered here. The connection (16) is a building block of the decoherence functional for relativistic particles, using which we may construct a field-theoretic description. The quantum phase it introduces in the decoherence functional coincides with the U(1) phase of the fiber bundle $\lambda_A \rightarrow I^\mu = \lambda^* \sigma^\mu \lambda$. This phase is a *classical*, geometric object appearing naturally in the lightcone's structure. On the other hand the quantum phases appearing in the decoherence functional are objects that may be measured statistically in interference patterns. Hence apparently the same mathematical object appears in two distinct physical roles. As the geometric phase is the main reason for the introduction of complex numbers in quantum theory, this situation brings to mind Penrose's conjecture about a relation between the complex numbers in quantum theory and the lightcone geometry. If anything the considerations in this paper lend plausibility to this idea.

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